Chaos Control and Modified Projective Synchronization of Chaotic Dissipative Gyroscope Systems

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Abstract: This paper proposes the chaos control and the modified projective synchronization methods for chaotic dissipative gyroscope systems. Because of the nonlinear terms of the gyroscope system, the system exhibits chaotic motions. Occasionally, the extreme sensitivity to initial states in a system operating in chaotic mode can be very destructive to the system because of unpredictable behavior. In order to improve the performance of a dynamic system or avoid the chaotic phenomena, it is necessary to control a chaotic system with a periodic motion beneficial for working with a particular condition. As chaotic signals are usually broadband and noise like, synchronized chaotic systems can be used as cipher generators for secure communication. This paper presents chaos synchronization of two identical chaotic motions of symmetric gyroscopes. Using the variable structure control technique, control laws are established which guarantees the chaos control and the modified projective synchronization. By Lyapunov stability theory, control laws are proposed to ensure the stability of the controlled and synchronized system. Numerical simulations are presented to verify the proposed control and the synchronization approach. This paper demonstrates that synchronization and anti-synchronization can coexist in dissipative gyroscope systems via variable structure control.

Key words: Dissipative gyroscope, chaos control, modified projective synchronization, variable structure control.

1. Introduction

Chaos in control systems and controlling chaos in dynamical systems have both attracted much interest in recent years. A chaotic system has complex dynamical behaviors that possess some special features, such as excessive sensitivity to initial conditions, broad spectrums of Fourier transform, bounded and fractal properties of the motion in the phase space, etc..

Chaotic phenomena can be found in many scientific and engineering fields such as biological systems, electronic circuits, power converters, chemical systems, and etc. [1]. The pioneering work of Ott, Grebogi, and Yorke [2] proposed the well-known OGY control method, where the control of chaotic systems has been widely studied.

Chaos control can be mainly divided into two categories [3]: one is the suppression of the chaotic dynamical behavior and the other is to generate or enhance chaos in nonlinear systems. Nowadays, different techniques have been proposed to achieve chaos control [4-6].

Also, since the synchronization of chaotic dynamical systems has been observed by Pecora and Carroll [7] in 1990, chaos synchronization has become a topic of great interest [8-10]. Synchronization phenomena have been reported in the recent literature. Until now, different types of synchronization have been found in interacting chaotic systems, such as complete synchronization [7], generalized synchronization [11], phase synchronization [12] and anti-phase
synchronization [13], etc.. In 1999, projective synchronization has been first reported by Mainieri and Rehacek [14] in partially linear systems that the master and slave vectors synchronize up to a constant scaling factor $\alpha$ (a proportional relation). Later, some researchers have extended synchronization to a general class of chaotic systems without the limitation of partial-linearity, such as non-partially-linear systems [15, 16]. After that, a new synchronization, called generalized projective synchronization, has been observed in the chaotic systems [17-19].

Recently, Li [20] considered a new synchronization method, called modified projective synchronization (MPS), where the responses of the synchronized dynamical states synchronize up to a constant scaling matrix. Therefore, the synchronization and anti-synchronization can coexist in a chaos synchronization problem.

The dynamics of a gyro is a very interesting nonlinear problem in classical mechanics. The concept of chaotic motion in a gyro was first presented in 1981 by Leipnik and Newton [21], showing the existence of two strange attractors. Recently, many researches address the stability, nonlinear phenomena and technical applications of the gyroscopic system [22, 23]. Gyros for sensing angular motion are used in airplane automatic pilots, rocket-vehicle launch-guidance, space-vehicle attitude systems, ship’s gyrocompasses and submarine inertial auto-navigators. Some methods have been presented for the control and synchronization of the nonlinear gyro system [24, 25].

The variable structure control (VSC) technique is a discontinuous control strategy that involves, first, selecting a sliding surface for the desired dynamics and, secondly, designing a discontinuous control law such that the system trajectory first reaches the surface and then stays in it forever [26]. Very recently, several researchers have used the sliding mode control technique to control chaotic systems [27-31].

In this paper, the chaos control and modified projective synchronization (MPS) for chaotic dissipative gyroscope systems are considered. To achieve these objectives, the VSC has been proposed. Control laws for chaos control and MPS of gyroscope are obtained by VSC, which is proved by the Lyapunov stability theory.

This paper is organized as follows. In section 2, the dynamics of a chaotic dissipative gyroscopic system is explained. The chaos control problem and VSC design to achieve this goal are described in section 3. Also, a lemma and a theorem for chaos control of dissipative gyroscopic system via VSC are proved in this section. In section 4, the MPS problem and VSC design to achieve this goal are described. Also, a theorem for MPS of dissipative gyroscopic systems via VSC is proved in sections 4. Finally, to show the effectiveness of the proposed control method for chaos control and MPS of gyroscopic systems, simulations are presented in section 5. At the end, the paper is concluded in section 6.

2. Nonlinear Dissipative Gyro Dynamics

The gyroscopic contains a mechanical vibration absorber in the interior in the form of a spring-mass-dashpot. The absorber mass (m) is centered on the z axis and position parallel to z axis. The spring has constant k, and dashpot has damping constant C. The geometry problem under consideration is depicted in Fig. 1. The motion of symmetric gyroscopic mounted on a vibrating base can be described by Euler’s angels $\theta$, $\phi$ and $\psi$. The Lagrangian can be expressed as [23]:

$$L = \frac{1}{2} \left( I_1 + m z^2 \right) \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta + \dot{\psi} \cos \theta \right) + \frac{1}{2} I_3 \dot{\psi} \cos \theta - \frac{k}{2} (z - h_0)^2$$

where $I_1$ and $I_3$ are the polar and equatorial moments of inertia of the typical gyroscopic, $M_g$ is the gravity force, $T$ is the amplitude of the external excitation disturbance, $\omega$ is the frequency of the external excitation disturbance, m is the mass of damper, and k
is the spring constant. It is clear that coordinates $\theta$ and $\phi$ are cyclic, which provides the conjugate momentum. The momentum integrals are $\beta_\theta$ and $\beta_\phi$.

Let the state variables $x = [\theta \; \phi \; z - p \; \dot{z}]^T$, then the dynamic equations can be rewritten as:

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{\beta_\phi^2}{[I_z + m(x_3 + p^2)]} \left(1 - \cos x_1\right) \\
&+ \frac{\left[ M_g + m_g p \right] + m_x x_3 + (M + m_x) g}{I_z + m(x_3 + p^2)} \sin \omega t \sin x_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\beta_\phi^2}{[I_z + m(x_3 + p^2)]} \left(1 - \cos x_1\right)^2 (x_3 + p) + g \left(1 - \cos x_1\right) \\
&- \frac{k}{m} x_3 + (x_3 + p) \dot{x}_3^2 - 2c x_4
\end{align*}
$$

where $p$ is $I_y - \frac{mg}{k}$. The gyroscope system, Eq. (2), performs complex dynamics and has been extensively studied by Ge [23] and Yau [24]. With specific values set at $I_z = 1$, $k = 100$, $l = 0.1$, $M = 0.5$, $m = 0.1$, $p = 0.1$, $\beta_\phi = 100$, $\omega = 2$, $2c = 0.5$ and $T = 5$ in numeric simulation, the dynamic behavior gyrooscope system, Eq. (2), exhibits an irregular motion as shown in Figs. 2 and 3 with initial conditions of $(x_1, x_2, x_3, x_4) = (-1.2, 1, -1.1, 0.00905)$.

Figs. 2 and 3 show that the gyroscope system trajectories are in a state of chaotic motion.

In the next section, the chaos control problem of dissipative gyroscope is described.

3. Chaos Control Problem of Dissipative Gyros via Variable Structure Control

In the previous section, it has been shown that the heavy symmetric gyro considered exhibits chaotic motion. The extreme sensitivity to initial states in a system operating in chaotic mode can be very destructive to the system because of unpredictable behavior. Sometimes, chaos is unwanted or undesirable.

In order to improve the performance of a dynamic system or avoid the chaotic phenomena, we need to control a chaotic system with a periodic motion which is beneficial for working with a particular condition.

It is thus of great practical importance to develop suitable control methods. Many researchers have been focused on this type of problem controlling chaos. Anti-control of chaos is interesting, nontraditional, and very challenging [32, 33].
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For this purpose, in this section, variable structure control is used as anti-control to chaos control of heavy symmetric gyroscope.

3.1 Chaos Control Problem

Now, control inputs are introduced in the Eq. (2) for the second and third states. Thus, the controlled nonlinear gyro becomes as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \left(\frac{\beta_y}{l + m x_1 + p x_1} \right) (1 - \cos \beta) \\
&+ \left[ (M_i + m_t) x_1 + (M + m) T \sin \alpha \sin \beta \right] u(t) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \left(\frac{\beta_y}{l + m x_1 + p x_1} \right) (1 - \cos \beta) (x_1 + p) + g(1 - \cos \beta) \\
&- \frac{1}{m} \left( x_3 + (x_1 + p) \right) x_3^2 - 2c x_4 + u_2(t)
\end{align*}
\]

(3)

where \( u_1, u_2 \in \mathbb{R} \) are the control inputs attached in the nonlinear gyroscopic system.

In order to simplify the following procedure, two nonlinear functions are defined as follows:

\[
\begin{align*}
f_1(x_1, x_3) &= \left(\frac{\beta_y}{l + m x_1 + p x_1} \right) (1 - \cos \beta) \\
&+ \left[ (M_i + m_t) x_1 + (M + m) T \sin \alpha \sin \beta \right] u(t) \\
f_2(x_1, x_2, x_3, x_4) &= \left(\frac{\beta_y}{l + m x_1 + p x_1} \right) (x_1 + p) \\
&+ g(1 - \cos \beta) - \frac{1}{m} \left( x_3 + (x_1 + p) \right) x_3^2 - 2c x_4
\end{align*}
\]

(4) (5)

The control problem is to drive the system to track a four-dimensional desired vector \( X_d(t) \) as follows:

\[
X_d(t) = \left[ x_{d,1}, x_{d,2}, x_{d,3}, x_{d,4} \right]^T
\]

(6)

which belongs to a class of C function on \([t_0, \infty)\). Let us define the tracking error as:

\[
E(t) = \left[ e_1(t), e_2(t), e_3(t), e_4(t) \right]^T
\]

(7)

Then, the error dynamics can be obtained from Eqs. (8) and (3)-(5) as follows:

\[
\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= f_1(x_1, x_3) - \dot{x}_{d,1} + u_1(t) \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= f_2(x_1, x_2, x_3, x_4) - \dot{x}_{d,3} + u_2(t)
\end{align*}
\]

(8)

The control goal considered in this section is that for any given target orbit \( X_d(t) \), the controller is designed such that the resulting tracking error vector satisfies:

\[
\lim_{t \to \infty} \| E(t) \| = 0
\]

(9)

where \( \| \cdot \| \) is the Euclidean norm of a vector.

3.2 Switching Surface and VSC Design

Using the VSC control method to control the chaotic gyroscope system involves two basic steps:

1. Selecting an appropriate sliding surface such that the sliding motion on the sliding manifold is stable and ensures \( \lim_{t \to \infty} \| E(t) \| = 0 \).

2. Establishing a robust control law which guarantees the existence of the sliding manifold \( S(t) = 0 \). The sliding surfaces are defined as \([34]\):

\[
\begin{align*}
S_1(t) &= e_2(t) + \delta \dot{e}_1(t) \\
S_2(t) &= e_4(t) + \delta \dot{e}_3(t)
\end{align*}
\]

(10) (11)

where \( \delta(t) \in \mathbb{R} \) and \( \delta \) is a real positive constant. Differentiating Eqs. (10) and (11) with respect to time as follow:

\[
\begin{align*}
\dot{S}_1(t) &= \dot{e}_2(t) + \delta \ddot{e}_1(t) \\
&= \dot{f}_1(x_1, x_2) - \ddot{x}_{d,1} + u_1(t) + \delta \dot{e}_2(t) \\
\dot{S}_2(t) &= \dot{e}_4(t) + \delta \ddot{e}_3(t) \\
&= \dot{f}_2(x_1, x_2, x_3, x_4) - \ddot{x}_{d,3} + u_2(t) + \delta \dot{e}_4(t)
\end{align*}
\]

(12) (13)

The rate of convergence of the sliding surface is governed by the value assigned to parameter \( \delta \).

Having established appropriate sliding surfaces, the next step is to design VSC to drive the error system trajectories onto the sliding surfaces. Before stating the scheme of the controller, the reaching condition of the sliding mode is given below.

Lemma 1. The motions of the sliding surfaces (10), (11) are asymptotically stable, if the following reaching condition is satisfied:
\[ S_{(i+1,2)}(t) \cdot \dot{S}_{(i+1,2)}(t) < 0 \]  

(14)

**Proof.** Let us define the Lyapunov function as:

\[ V(t) = \frac{1}{2} S_1^2(t) + \frac{1}{2} S_2^2(t) \]  

(15)

According to Lyapunov stability theory, condition Eq. (14) ensures that

\[ V(t) = \sum_{i=1}^{2} S_i(t) \cdot \dot{S}_i(t) < 0 \]  

(16)

Then, \( S_{(i+1,2)}(t) \) are the switching surfaces, Eqs. (10) and (11) are asymptotically stable.

The current variable structure controller design is described as follows:

**Theorem 1:** Consider the chaos control problem represented by Eq. (8). If the control inputs \( u_1(t) \) and \( u_2(t) \) are suitably designed as:

\[ u_1(t) = -\eta_1 \text{sign}(S_1(t)) - f_1(x_1, x_2) + \ddot{x}_{d,1} - \delta_1 \epsilon_1(t) \]  

(17)

\[ u_2(t) = -\eta_2 \text{sign}(S_2(t)) - f_2(x_1, x_2, x_3, x_4) + \ddot{x}_{d,2} - \delta_2 \epsilon_2(t) \]  

(18)

where \( S_1(t) = \epsilon_1(t) + \delta_1 \epsilon_1(t) \), \( S_2(t) = \epsilon_2(t) + \delta_2 \epsilon_2(t) \) and \( \delta_{(i+1,2)}, \delta_{(i+1,2)} \) are positive constant parameters.

Then the hitting condition Eq. (14) of the sliding mode is satisfied, and the trajectories of error dynamics will converge to the sliding surfaces \( S_{(i+1,2)}(t) = 0 \).

**Proof 1:** Define two switching surfaces as:

\[ \dot{S}_1(t) = e_1(t) + \delta_1 \epsilon_1(t) \]  

(19)

\[ \dot{S}_2(t) = e_2(t) + \delta_2 \epsilon_2(t) \]  

(20)

Differentiating Eqs. (19) and (20) with respect to time as follow:

\[ \dot{\dot{S}}_1(t) = \ddot{e}_1(t) + \delta_1 \dot{\epsilon}_1(t) \]  

(21)

\[ \dot{\dot{S}}_2(t) = \ddot{e}_2(t) + \delta_2 \dot{\epsilon}_2(t) \]  

(22)

Define a Lyapunov function as:

\[ V(t) = \frac{1}{2} S_1^2(t) + \frac{1}{2} S_2^2(t) \]  

(23)

Differentiating Eq. (23) with respect to time, we have:

\[ \dot{V}(t) = S_1(t) \dot{S}_1(t) + S_2(t) \dot{S}_2(t) \]  

(24)

Substituting Eqs. (21) and (22) into Eq. (24):

\[ \dot{V}(t) = S_1(t) [\ddot{e}_1(t) - \ddot{x}_{d,1} + u_1(t) + \delta_1 \epsilon_1(t)] \]

\[ + S_2(t) [\ddot{e}_2(t) - \ddot{x}_{d,2} + u_2(t) + \delta_2 \epsilon_2(t)] \]  

(25)

Let

\[ \dot{u}_1(t) = -\eta_1 \text{sign}(S_1(t)) - f_1(x_1, x_2) + \ddot{x}_{d,1} - \delta_1 \epsilon_1(t) \]  

(26)

and

\[ \dot{u}_2(t) = -\eta_2 \text{sign}(S_2(t)) - f_2(x_1, x_2, x_3, x_4) + \ddot{x}_{d,2} - \delta_2 \epsilon_2(t) \]  

(27)

where \( \eta_{(i+1,2)} \) are positive constant parameters. Then

\[ \dot{V}(t) = -\eta_1 |S_1(t)| - \eta_2 |S_2(t)| \]  

(28)

The reaching condition \( \dot{S}_1 < 0 \) is always satisfied. Furthermore, according to Lemma 1, \( S_{(i+1,2)}(t) \to 0 \).

Thus, Theorem 1 is proven.

4. Chaos Synchronization Problem of Dissipative Gyros via VSC

4.1 Modified Projective Synchronization Problem

Consider two coupled, chaotic dissipative gyro systems, where the master and slave systems are denoted by \( x \) and \( y \), respectively. The master system is shown in Eq. (2) and the slave system is shown in Eq. (3), notice that in this equation \( x \) has to be denoted by \( y \).

In order to simplify the following procedure, two nonlinear functions are defined as follows:

\[ g(x, z) = -\frac{\rho_1^2}{[l_1 + m(z + p^2)]^2 \sin^2 x} \]  

(29)

\[ h(x, y, v) = \frac{\rho_2^2}{[l_1 + m(z + p^2)]^2 \sin^2 x} (z + p) \]  

(30)

Defining the synchronization errors between the master and slave systems as follows:

\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3 \\
\dot{e}_4
\end{bmatrix} =
\begin{bmatrix}
y_1 - x_1 \\
y_2 - x_2 \\
y_3 - x_3 \\
y_4 - x_4
\end{bmatrix} -
\begin{bmatrix}
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 \\
0 & 0 & \alpha_2 & 0 \\
0 & 0 & 0 & \alpha_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\]

(31)

where \( \alpha_1, \alpha_2 \in R \) are the scaling factors that define a proportional relation between the synchronized systems.

Obviously, the modified projective synchronization is defined as the generalized projective synchronization if \( \alpha_1 \) is equal to \( \alpha_2 \).

The error dynamics can be obtained as:
\[
\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= (1-\alpha_1)g(x,z) + u_1(t) \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= (1-\alpha_2)h(x,y,z,v) + u_2(t)
\end{align*}
\] (32)

The objective of the current control problem is to design the appropriate control signals \(u_1(t)\) and \(u_2(t)\) such that for any initial conditions of the master and slave systems, the synchronization errors converge to zero such that the resulting synchronization error vector satisfies Eq. (9).

In the next section, the controllers are designed to achieve this objective.

### 4.2 Switching Surface and VSC Design for MPS

Using the VSC control method to MPS the chaotic gyroscope system involves two basic steps described in the section 3.2.

To solve this problem, let us to differentiate Eq. (10) and (11) with respect to time as follow:

\[
\begin{align*}
\dot{S}_i(t) &= \dot{e}_i(t) + \delta_i \dot{e}_i(t) \\
&= (1-\alpha_i)g(x,z) + u_i(t) + \delta_i e_i(t) \\
\dot{S}_2(t) &= \dot{e}_4(t) + \delta_2 \dot{e}_4(t) \\
&= (1-\alpha_2)h(x,y,z,v) + u_2(t) + \delta_2 e_4(t)
\end{align*}
\] (33, 34)

The following theorem shows the properties of the MPS chaotic dissipative gyroscope systems via VSC.

**Theorem 2:** Consider the MPS problem represented by Eq. (32). If the control inputs \(u_1(t)\) and \(u_2(t)\) are suitably designed as:

\[
\begin{align*}
u_1(t) &= -\eta_1 \text{sign}(S_1(t)) - (1-\alpha_1)g(x,z) - \delta_2 e_2(t) \\
u_2(t) &= -\eta_2 \text{sign}(S_2(t)) - (1-\alpha_2)h(x,y,z,v) - \delta_2 e_4(t)
\end{align*}
\] (35, 36)

where \(S_1(t) = e_2(t) + \delta_2 e_2(t)\), \(S_2(t) = e_4(t) + \delta_2 e_4(t)\) and \(\eta_{1(\alpha_1)}\), \(\eta_{2(\alpha_2)}\) are positive constant parameters.

Then the hitting condition Eq. (14) of the sliding mode is satisfied, and the trajectories of error dynamics will converge to the sliding surfaces \(S_{1(\alpha_1)}(t) = 0\).

**Proof 2:**

Define two sliding surfaces as:

\[
\begin{align*}
S_1(t) &= e_2(t) + \delta_2 e_2(t) \\
S_2(t) &= e_4(t) + \delta_2 e_4(t)
\end{align*}
\] (37, 38)

Differentiating Eqs. (37) and (38) with respect to time as follow:

\[
\begin{align*}
\dot{S}_1(t) &= \dot{e}_2(t) + \delta_2 \dot{e}_2(t) \\
&= (1-\alpha_1)g(x,z) + u_1(t) + \delta_2 e_2(t) \\
\dot{S}_2(t) &= \dot{e}_4(t) + \delta_2 \dot{e}_4(t) \\
&= (1-\alpha_2)h(x,y,z,v) + u_2(t) + \delta_2 e_4(t)
\end{align*}
\] (39, 40)

Define a Lyapunov function as:

\[
V(t) = \frac{1}{2} S_1^2(t) + \frac{1}{2} S_2^2(t)
\] (41)

Differentiating Eq. (41) with respect to time, we have:

\[
\dot{V}(t) = S_1(t) \dot{S}_1(t) + S_2(t) \dot{S}_2(t)
\] (42)

Substituting Eqs. (39) and (40) into Eq. (42):

\[
\dot{V}(t) = S_1(t) \left[ (1-\alpha_1)g(x,z) + u_1(t) + \delta_2 e_2(t) \right] \\
+ S_2(t) \left[ (1-\alpha_2)h(x,y,z,v) + u_2(t) + \delta_2 e_4(t) \right]
\] (43)

Let

\[
\begin{align*}
u_1(t) &= -\eta_1 \text{sign}(S_1(t)) - (1-\alpha_1)g(x,z) - \delta_2 e_2(t) \\
u_2(t) &= -\eta_2 \text{sign}(S_2(t)) - (1-\alpha_2)h(x,y,z,v) - \delta_2 e_4(t)
\end{align*}
\] (44, 45)

where \(\eta_{1(\alpha_1)}\), \(\eta_{2(\alpha_2)}\) are positive constant parameters. Then

\[
\dot{V}(t) = -\eta_1 |S_1(t)| - |S_2(t)|
\] (46)

The reaching condition \((SS < 0)\) is always satisfied. Furthermore, according to Lemma.1, \(S_{1(\alpha_1)}(t) \to 0\) . Thus, Theorem 2 is proven.

### 5. Simulation Results

In this section, numerical simulations are given to demonstrate chaos control and MPS of the chaotic dissipative gyros via VSC. The parameters of dissipative gyros systems are specified as follows:

\[
\begin{align*}
l &= 1 \quad , \quad k = 100 \quad , \quad l = 0.1 \quad , \quad M = 0.5 \quad , \quad m = 0.1 \quad , \quad p = 0.1 \quad , \\
\beta^2 &= 100 \quad , \quad \omega = 2 \quad , \quad \varepsilon = 0.5 \quad , \quad T = 5
\end{align*}
\]

which, as shown in section 2, give rise to a chaotic state.

To reduce the system chattering, the sign functions in VSC control are substituted with the saturation functions.

### 5.1 Simulation Results of Chaos Control

Numerical simulations are given to demonstrate tracking control of the dissipative gyro to two desired trajectories: regular and periodic trajectories. Regular trajectories are defined as follows:

\[
X_d_{\text{Regular}}(t) = \left[ \frac{\pi}{3}, 0, 0.5, 0 \right]^T = \left[ 1.047, 0, 0.5, 0 \right]^T
\]
And periodic trajectories are defined as follow:

\[
X_{d_{\text{Periodic}}}(t) = \begin{bmatrix}
A \cos(wt) + d_1 \\
-A \sin(wt) \\
A \sin(wt) + d_2 \\
A \cos(wt)
\end{bmatrix}
\]

where \( A = 1 \), \( w = 1 \) and \( d_1 = d_2 = 1.5 \).

The initial conditions are defined as \([x_1(0) \ x_2(0) \ x_3(0) \ x_4(0)]^T = [-1 \ 1 \ -1 \ 1]^T\).

The time responses of the dissipative gyroscope controlled to track regular trajectories are shown in Fig. 4, and the corresponding error states converge asymptotically to zero in Fig. 5. Also, the sliding surfaces converge asymptotically to zero in Fig. 6.

The time responses of the dissipative gyroscope controlled with track periodic trajectories are shown in Fig. 7, where the corresponding error states converge asymptotically to zero in Fig. 8. The sliding surfaces converge asymptotically to zero in Fig. 9.

The simulation results of chaos control of dissipative gyro via VSC have good performances and confirm that the error states are asymptotically regulated to zero.

5.2 Simulation Results of MPS

Numerical simulations are given to demonstrate MPS of the chaotic dissipative gyros via VSC.
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Fig. 7  Time responses of state variables (tracking periodic trajectories): trajectories of state variables and desired states are shown with solid and dashed line, respectively.

Fig. 8  Time response of tracking error states (tracking periodic trajectories).

The scaling factor matrix is specified as:

\[
\begin{bmatrix}
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 \\
0 & 0 & \alpha_2 & 0 \\
0 & 0 & 0 & \alpha_2
\end{bmatrix} = \begin{bmatrix}
0.8 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0 \\
0 & 0 & 0.4 & 0 \\
0 & 0 & 0 & 0.4
\end{bmatrix}
\]

The initial conditions are defined as:

\[
[x_1(0)，x_2(0)，x_3(0)，x_4(0)]^T = [-1.2，1，-1.1，0.000905]^T \\
[y_1(0)，y_2(0)，y_3(0)，y_4(0)]^T = [0.5，0.5，0.5，0.5]^T
\]

The initial conditions are defined as:

\[
[x_1(0)，x_2(0)，x_3(0)，x_4(0)] = [1.2，1，1.1，0.000905] \\
[y_1(0)，y_2(0)，y_3(0)，y_4(0)] = [0.5，0.5，0.5，0.5]
\]

The time responses of controlled master-slave chaotic gyros are shown in Fig. 10a and Fig. 10b.

Obviously, the MPS errors converge asymptotically to zero in Fig. 11. Also, the sliding surfaces converge asymptotically to zero in Fig. 12. The phase plane of the master and the slave systems are shown in Fig. 13.

The simulation results of MPS via VSC have good performances and confirm that the master and slave systems achieve the modified projective synchronized states. Also, these results demonstrate that the system error states are asymptotically regulated to zero.

6. Conclusions

In this paper, the variable structure controls have been designed to chaos control and modified projective synchronization of chaotic dissipative gyroscope systems.

Suppression of the chaos is presented so as to improve the performance of a dynamical system.
Chaos Control and Modified Projective Synchronization of Chaotic Dissipative Gyroscope Systems

Fig. 10 (a) Time response of the master and the slave systems (with scaling factors $\alpha_1 = 0.8, \alpha_2 = 0.4$); (b) Time response of the slave and the proportional master systems (with scaling factors $\alpha_1 = 0.8, \alpha_2 = 0.4$). Trajectories of the slave and the proportional master systems are shown with solid and dashed line, respectively.

Fig. 11 Time response of MPS error states (with scaling factors $\alpha_1 = 0.8, \alpha_2 = 0.4$).

Fig. 12 The sliding surfaces of MPS (with scaling factors $\alpha_1 = 0.8, \alpha_2 = 0.4$).

To achieve MPS, it is clear that the proposed method is capable to create a full range MPS of all state variables in a proportional way. It also allows us to arbitrarily adjust the desired scaling by controlling the slave system.
The advantages of this method can be summarized as follows:

- It is a systematic procedure for the control and synchronization of dissipative gyroscope systems.
- The controllers are easy to be implemented.
- It is not necessary to calculate the Lyapunov exponents and the Eigen values of the Jacobian matrix, which makes it simple and convenient.
- Synchronization and anti-synchronization can coexist in dissipative gyroscope systems.
- Simulations results have verified the effectiveness of this method to control and synchronize chaotic gyroscopes.
- Since the gyro has been utilized to describe the mode in navigational, aeronautical or space engineering, the modified projective synchronization procedure in this study may have practical applications in the future.

References


