Teaching and learning how to write proofs in Concepts of Geometry∗

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Abstract: This paper concerns my experience teaching Concepts of Geometry, an inquiry-based course that emphasizes discovery learning, analytical thinking, and individual creativity. The author discusses how to guide the students to recognize the connections among different mathematical ideas; to select the types of reasoning and methods of proof that apply to the problem; to organize their mathematical thinking through communication; to make their own mathematical conjectures; and to analyze and evaluate each other’s mathematical arguments and proofs. In particular, the comments and the feedback from students are included.

Key words: inquiry based teaching; discovery learning; constructivist based approach

The Mathematics Association of American’s Committee on the Undergraduate Program in Mathematics (CUPM) made recommendations to guide mathematics departments in designing curricula for their undergraduate students. The following statement is quoted from the CUPM Curriculum Guide 2004:

Recommendation 2: Develop mathematical thinking and communication skills.
Every course should incorporate activities that will help all students’ progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind. More specifically, these activities should be designed to advance and measure students’ progress in learning to:

(1) State problems carefully, modify problems when necessary to make them tractable, articulate assumptions, appreciate the value of precise definition, reason logically to conclusions, and interpret results intelligently;
(2) Approach problem solving with a willingness to try multiple approaches, persist in the face of difficulties, assess the correctness of solutions, explore examples, pose questions, and devise and test conjectures;
(3) Read mathematics with understanding and communicate mathematical ideas with clarity and coherence through writing and speaking.

In responding to the CUPM Curriculum Guide, the author started a project to use an inquiry-based teaching style in Concepts of Geometry, a junior- and senior-level college geometry class. This class emphasizes discovery learning, analytical thinking and individual creativity. The goal was to guide the students to recognize the connections among different mathematical ideas; to select the types of reasoning and methods of proof that apply to the problem; to organize their mathematical thinking through communication; to make their own mathematical conjectures; and to analyze and evaluate each other’s mathematical arguments and proofs.

In this report the author present the results of a study investigating the effectiveness of teaching proof writing via an inquiry-based teaching style in his Concepts of Geometry class during a two-year period, Fall 2006-Fall 2007.

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The study involved developing inquiry-based teaching/learning material, a set of survey questions for the students, and the analysis of the answers received. According to the survey, the author found a wide variation in the responses from the students during the period of the study with the trend toward favoring an inquiry-based teaching approach.

This study suggests that using an inquiry-based teaching approach does indeed improve learning, in particular, learning to write proofs.

1. Introduction

Geometry is an important component of the K-12 curriculum and is required by the Department of Education. As part of the requirements to complete their undergraduate degrees, pre-service secondary mathematics teachers (PSMTs) complete many mathematics courses, including a geometry course. The textbook we use is *Roads to Geometry* by Wallace and West. This book is for an upper-level class, and it clarifies, extends, and unifies concepts discussed in basic high school geometry courses. Topics include axiomatic systems, axiom sets for geometry, neutral geometry, Euclidean geometry of the plane, analytic and transformational geometry, non-Euclidean geometries and projective geometry. A key characteristic of the book is proof writing. Writing proofs is an extremely difficult mathematical concept for students. Studies such as Senk (1985), Moore (1994), and Weber (2001) have shown that many students who completed the proof-oriented courses, such as high school geometry, introduction to proof, and abstract algebra, are unable to construct anything beyond very trivial proofs. Moreover, they cannot determine whether a proof is valid or not.

After teaching *Concepts of Geometry* via traditional lecture for four years, the author noticed that most of the time when the author introduced proof writing or constructed proofs, the students passively copied the lecture notes without really understanding the material. They did not learn how to reason; therefore, they did not know how to construct a proof by themselves. Furthermore, the students had great difficulty in following the proofs written in the textbook. The author began to wonder whether or not an inquiry-based teaching approach would have a positive effect on students’ learning of proof writing in geometry class.

Seymour wrote, “You can’t teach people everything they need to know. The best you can do is position them where they can find what they need to know when they need to know it”. Discovery or inquiry-based teaching has proven to be far more effective than the traditional lecture approach in the classroom according to the current research in learning theory. The concept of discovery learning has appeared numerous times throughout history as a part of the educational philosophy of many great educators. Discovery learning has shown that if provided a personal and constructive learning environment, students will lead themselves to discover the truth through learned experiences and prior knowledge.

However, the traditional lecture approach plays a key role in all classes in the department of mathematics at the author’s university, Southeast Missouri State University (Southeast). Although most faculty members in the department are aware of the inquiry teaching method, it has not received overwhelming acceptance due to many obstacles. The major obstacles are: (1) lack of experience; (2) lack of supporting material; (3) lack of administration support; and (4) lack of peer interest.

The Educational Advancement Foundation (EAF) is an organization that supports the development and implementation of inquiry-based learning at all educational levels in the United States, particularly in the fields of mathematics and science. Moreover, the Center for Scholarship in Teaching and Learning at our institution offers
assistance and small grants to faculty to study how their pedagogy is linked to learning. In the year 2006, the author was awarded a grant to establish a pilot project for an inquiry-based learning mentoring program in the Department of Mathematics at Southeast Missouri State University. We developed inquiry-based learning material (a problem sequence) suitable for the three-credit-hour *Concepts of Geometry* course in the summer of 2006 and used the material in the Fall 2006 *Concepts of Geometry* class. The author taught the *Concepts of Geometry* class for the first time in Fall 2006. In Spring 2007, the author was awarded the Center for Scholarship in Teaching and Learning (SoTL) grant and became a SoTL fellow, which allowed me to continue the pilot project in the *Concepts of Geometry* class through the Fall 2007. This project was designed to investigate the effectiveness of using an inquiry-based teaching approach to enhance students’ learning proof writing.

2. Method

First, before the fall semester started, we developed inquiry-based teaching-learning material and a problem sequence, including material in both Euclidean and non-Euclidean geometry. This material was designed to help all students develop a deep understanding of mathematical concepts and how to apply them. It also prepares them for using their newfound skills to further their education or the jobs. Each section of the material starts with basic definitions and examples, then moves to lemmas, theorems, and corollaries, and ends with exercises and reflection questions. It challenges students to explore open-ended situations actively. The aim of using the material is to help students understand basic background knowledge, routinely investigate specific cases, look for and articulate patterns, and make, test, and prove conjectures. The problem sequence was provided to the students free of charge at the beginning of the semester.

Second, the author discussed his teaching progress with his mentor in Fall 2006, attended SoTL fellow monthly meetings, and communicated with mentors and other SoTL fellows to exchange teaching experience and ideas to improve his teaching in the classroom in Fall 2007.

Third, in my daily classroom teaching, the author arranged opportunities for student presentations, observed students’ performance and recorded their progress. The purpose was to enable the students to learn more by discovering more for themselves. Typically, a lesson would begin with the giving definitions; then the students would explain the meaning of the definitions, draw the geometric figures, and provide examples. Then, the students built the rest of the body of knowledge as problems and proofs were assigned. Collaboration with classmates was encouraged. Students with mixed abilities were teamed together to work on group activities; each group made its own group work report. As the teacher, the author was available to assist students, providing hints to the problems or the proofs.

Fourth, the author provided classroom activities and projects that use hands-on learning. In particular, the author assigned a project that required the students to construct geometric objects, explain the geometry involved in their constructions, make appropriate conjectures, and possibly prove their conjectures.

Fifth, the author gave periodic survey questions to obtain students’ feedback and to monitor their learning. Moreover, the author used answers to the survey questionnaires to analyze the effectiveness of his teaching and their learning. The author compared the final examination score, the numerical course average, and the course letter grade for each student in the experimental section with the average final examination score, course average, and course letter grade of students in the section taught with the traditional lecturing approach. However, the author did not use the test scores for statistical analysis for significant differences between the different teaching approaches since he had only eight students.
3. Observations on students’ progress

Throughout the semesters, the author observed the students’ progress. First, the author identified the students with different mathematical backgrounds, and the author identified the level of their understanding of proofs.

At the beginning of each semester, the students did not understand the statement of a theorem; they did not know the condition or the conclusion. In particular, the students had great difficulty in identifying the meaning of equivalence, and if and only if. They did not know what was given, and what should be proved. It was very common for the students to construct a proof with flaws in logic, and sometimes they could not even follow the proofs in the textbook.

After the first five weeks, the students started to pick up material on their own regarding Euclidean geometry. According to the author’s observation, and also from the survey, they felt comfortable with Euclidean geometry since they learned the basics in high school, and were familiar with the material. During the six-week study on Euclidean geometry, the author noticed that even the weaker students could prove some theorems of moderate difficulty. Although their approach was lengthy, they were able to convince their fellow students with their logic. On the other hand, the stronger students were able to provide a shorter and more concise proof. These students were able to communicate their mathematical thoughts clearly to one another. The weaker students contributed more and more in the group discussions and activities compared to their involvement during the first five weeks. This resulted in their homework and test grades being greatly improved. One difficulty that students encountered in Euclidean geometry was to construct auxiliary lines for more complicated problems. Although, sometimes, the stronger students could come up with some ideas, all students felt overwhelmed by these constructions.

The last six weeks of study were focused on non-Euclidean geometry. This new topic was not well received by the students at first, but as time went by they gradually understood the material and were able to solve problems and prove theorems in this setting.

Also in the last six weeks, the students presented their projects during the class period. Some of these class projects were good enough to be presented in the Annual Show Me Undergraduate Math Conference which was held by the Mathematics Department at Southeast and supported by the Mathematics Association of America in November 2006 and 2007. Some of the projects were quite good. For example, in Fall 2007, two students made individual presentations on soccer goalkeeping and pool shooting with respect to geometry and angles. Both of the students studied basic principles that lead to more efficient goalkeeping. Another student designed a fractal similar to the Sierpinski triangle, formulated a conjecture on the area of the triangle, and used mathematical induction to prove her conjecture. Also, one student studied projective geometry, did some undergraduate research independently, and submitted his research paper to a journal to be considered for publication.

4. Summary of survey questions

Assessment data was obtained from students in *Concepts of Geometry* in Fall 2006 and Fall 2007. The survey questions that were designed for the class were answered by each student in class. A summary of the students’ responses to the reflective questions designed for this project are presented below:

(1) Question 1: Classify all the theorems you learned and indicate the percentage of the theorems you feel were: (a) very difficult, (b) difficult, (c) average, (d) easy, (e) very easy.

Answer 1:
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There are different percentages according to students’ background, but in general, students considered 35% of the theorems very difficult, 10% difficult, 40% average, 5% easy, and 10% very easy.

(2) Question 2: What is the most difficult thing about this class?
Answer 2:
(a) The most difficult thing is deriving the proofs on your own.
(b) Struggled greatly with proofs, how to get from one point to another point.
(c) Determine what strategy to use in the beginning, whether it should be a proof by cases, contradiction, or start off with a new construction.
(d) Applying previously learned theorems when proving new ones.
(e) Getting a picture of the theorem and knowing what theorems should be used in the proof.
(f) Abstract thinking in Neutral Geometry.

(3) Question 3: Which is easier, neutral geometry or Euclidean geometry? Indicate your reasons.
Answer 3:
(a) Mostly Euclidean geometry is easier since it is most familiar geometry. But some theorems and proofs are difficult.
(b) Triangles and circles in Euclidean geometry can be easily visualized and drawn. The hardest theorems are the ones about quadrilaterals because it seems like there are more things to look at while trying to figure out what part of the information is the most useful to proving the theorems or problems.
(c) Neutral geometry was a new concept and it is more difficult. To prove theorems without using Euclid’s Parallel Postulates is difficult, but it is interesting.

(4) Question 4: How much did you learn in all aspects such as understanding the definition, knowing what the given is, knowing what the conclusion is, starting a proof, choosing direct proof or contradiction, constructing lines to assist proofs, etc.? List the most difficult one and the easiest one, and indicate why so.
Answer 4:
(a) Learned a great deal in this class. Understanding the definitions and difference between the three geometries, I have learned a lot and find this the easiest.
(b) I understand the general underlying concepts, but do not understand how to apply the known theorem to prove a new theorem. The hardest is starting the proof and finding where to begin without helpful hints.
(c) The easiest thing is knowing what is given and what is the conclusion. Although I know what I need to prove, I do not know how to get there.
(d) I learned the most when it comes to identifying theorems. Constructing lines to assist proof was the most difficult thing.
(e) Choosing direct or indirect proof was easiest. I am beginning to fully understand proofs. In each proof, I first indicate what is given and what I am trying to conclude to give the proof direction.
(f) Sometimes starting a proof is difficult in deciding whether to use a direct proof or one by contradiction, but once I get started, I can usually figure out a way to finish the proof. I am having fewer problems now that I have become more familiarized with the theorems.

(5) Question 5: How do you like the course materials we used? Does the sequence of the material aid your learning? Can you use the previous theorems to prove the next theorems?
Answer 5:
(a) The class material is very helpful. The material is in a good order. We can use the previously learned
material to prove the theorems later. The book is sometimes hard to understand.

(b) The sequence does help when we are learning the proofs.

(c) The course material is very helpful. The lists of theorems and definitions helps when proving new theorems because I can look back on what has already been proven or defined to assist in the new proof.

(d) I am starting to like the handouts more because they are in a logical sequence.

(6) Question 6: As a student, do you feel you learned in this class? If you were a teacher, what would you have done differently?

Answer 6:

(a) I did learn a lot from this class. I never thought about geometry in an abstract frame of mind. It is difficult, but I am learning.

(b) It is definitely the class in which I learned the most new things. It was the only class that I took this semester which I have never taken before in Germany. And therefore I learned some new theorems and I even learned a complete new geometry that I have never heard of before.

(c) As a student, I have learned a lot from this class. Once you break down all the theorems to prove them, it makes you look at other proofs in another way. I think this class will help me in the long run when I become a teacher, because we proved every theorem.

(d) I am satisfied with the knowledge that I am building through thinking through the proofs assigned for homework. I become familiar with the theorems of geometry and can apply them to real life situations.

(7) Question 7: If you were the teacher of this class, what instructional style would you use?

Answer 7:

(a) I enjoy the instructional style used in this course. It keeps the student engaged in learning by requiring the student to complete proofs on his own with definitions and examples given in class. It allows the student to be inspired to complete proofs on his own and gain experience with the material.

(b) I would use repetition. I like when I do things over and over to truly understand them.

(c) Using manipulative would aid my learning. It is beneficial to my study habits.

(d) Although it takes time to go through the proofs, it forces me to think through the theorem myself and gain a better understanding of each theorem.

5. Conclusion

In the author’s traditional lecture style geometry class, the students simply memorize the proofs without clear understanding. In this inquiry-based class, the student either solved problems by himself/herself, or was guided by the instructor without being provided with the explicit method. This strategy is very demanding on the instructor, as the instructor must know the material extremely well in order to guide the students to a solution and effectively correct missteps along the way. It is very demanding for the students as well. It is significantly different from instructional strategies that students have typically encountered in their mathematics classes as it requires them to be much more active learners. This certainly moves students away from their “comfort zone” in the short term, but it has long-term benefits in that the students develop their own proofs and solutions and have truly experienced the manner in which mathematicians work. This helps the students to become mathematically independent.

Based on the material and data collected, the following conclusions can be made for students in this Concepts of Geometry:
(1) Students enjoy the lively classroom activities;
(2) Students feel that they were challenged and stimulated to learn;
(3) Students gain confidence in their own problem solving ability;
(4) Students make progress in becoming independent learners;
(5) From this project, the author concludes that the inquiry-based teaching approach is suitable for students who have: (a) good mathematical background; (b) good reading and comprehension ability; (c) independent learning ability; (d) logical thinking ability; and (e) persistence.

As an instructor, the effect of this project to the author is that he learned to be open to different teaching approaches, and to be committed to apply appropriate teaching approaches to meet the needs of his students.

References:

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We can find a similar situation in teaching Chinese: Students who have learned simplified characters feel it is difficult to study the equivalent traditional characters. On the other hand, students who have studied traditional characters can readily and easily learn simplified characters. My investigation shows that among the students who studied Chinese for more than two years, 43% can “write traditional characters and read simplified characters”, 28% can “write simplified characters and read traditional characters”. Thus it can be seen that the former is easier than the latter. This is because the strokes of traditional characters in the majority of cases already include the strokes of simplified characters. We may try the following concrete methods to teach students how to “write traditional characters and read simplified characters”:

First, teachers request strictly, especially in the beginners’ class, if teachers do not put forward unambiguous requirements, no matter whether “writing traditional characters and reading simplified characters” or the reverse, not all students will comply.

Second, it is important to motivate students to like traditional characters through developing their interest in calligraphy and in other cultural aspects. Evidence is that gradual promotion of an interest in culture and art is a good way to stimulate students’ interest in traditional characters.

Third, emphasizing the phonetic and ideographic aspects as well as sources to show the functions of traditional characters will promote students’ understanding of the structural principles of Chinese characters and enhance their ability for logical memorization.

Fourth, the instructor should introduce the experience of freely converting between traditional characters and simplified characters using computer input methods, in order to minimize students’ dread of traditional and simplified characters transformation.

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