Rearrangement Inequality and Chebyshev’s Sum Inequality on Positive Tensor Products of Orlicz Sequence Space with Banach Lattice

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Abstract: Let \( \varphi \) be an Orlicz function that has a complementary function \( \varphi^* \) and let \( l_\varphi \) be an Orlicz sequence space. We prove a similar version of Rearrangement Inequality and Chebyshev’s Sum Inequality in \( l_\varphi \otimes_F X \), the Fremlin projective tensor product of \( l_\varphi \) with a Banach lattice \( X \), and in \( l_\varphi \otimes_I X \), the Wittstock injective tensor product of \( l_\varphi \) with a Banach lattice \( X \).

Keywords: Rearrangement inequality, Chebyshev’s sum inequality, injective tensor product, projective tensor product, Orlicz sequence space

1. Introduction

Fremlin in [5, 6] and Wittstock in [17, 18] investigated the positive projective tensor product and the positive injective tensor product of Banach lattices respectively. Now they are called the Fremlin projective tensor product and the Wittstock injective tensor product. Orlicz spaces were introduced in [14] in 1932, and they are a generalization of the classical space \( L_p \). As a special case of \( L_p \), \( l_p \) can also be generalized to Orlicz sequence space \( l_\varphi \). In [8], Lai proved the Rearrangement Inequality in the Fremlin projective tensor product \( l_\varphi \otimes_F X \) and the Wittstock injective tensor product \( l_\varphi \otimes_I X \) for \( 1 \leq p < \infty \). In this paper, building on the same idea, we generalize the results of [8] to the Fremlin projective tensor product \( l_\varphi \otimes_F X \) and the Wittstock injective tensor product \( l_\varphi \otimes_I X \) of an Orlicz sequence space \( l_\varphi \) with a Banach lattice \( X \). Using the newly proved Rearrangement Inequality, we then can prove the Chebyshev’s Sum Inequality in these two positive tensor products.

In this paper all vector spaces are over the real numbers. If \( X \) is an ordered set, the usual order on \( X^N \) is defined by \( (x_i) \geq 0 \iff x_i \geq 0 \) for each \( i \in \mathbb{N} \). For a Banach space \( X \), \( X^* \) denotes its dual space and \( B_X \) denotes its closed unit ball. Elements of a fixed Banach space \( X \) will be denoted by letters \( u, x, y, z \), while the elements of a dual space \( X^* \) will be denoted by \( u^*, x^*, y^*, z^* \). For a Banach lattice \( X \), its positive cone will be denoted by \( X^+ \). For unexplained terminology we refer to the reference [16].

2. Orlicz Sequence Spaces

An Orlicz function is a continuous non-decreasing and convex function \( \varphi : [0, +\infty) \to [0, +\infty) \) such that \( \varphi(t) = 0 \) only at \( t = 0 \) and \( \lim_{t \to +\infty} \varphi(t) = +\infty \). For any Orlicz function \( \varphi \), the Orlicz sequence space \( l_\varphi \) is defined by

\[
l_\varphi = \{ a = (a_i) \in \mathbb{R}^N : \sum_{i=1}^{\infty} \varphi(|\lambda a_i|) < +\infty \text{ for some } \lambda > 0 \}.
\]

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The space $l_\varphi$, equipped with the norm
\[ \|a\|_{l_\varphi} = \inf\{\lambda > 0 : \sum_{i=1}^{\infty} \varphi(|a_i/\lambda|) \leq 1\}, \]
is a Banach space.

Every Orlicz function $\varphi$ has a right derivative $p$ and $\varphi(t) = \int_0^t p(u)du$, for $t \geq 0$. If $p$ satisfies $p(0) = 0$ and $\lim_{t \to +\infty} p(t) = +\infty$ (these restrictions exclude only the case that $l_\varphi = l_1$, which is when $\varphi(t)$ is equivalent to $t$), then the right inverse $q$ of $p$, $q(s) = \sup\{t : p(t) \leq s\}$ for $s \geq 0$, is a right continuous non-decreasing function such that $q(0) = 0$ and $q(s) > 0$ whenever $s > 0$. Define $\varphi^*(s) = \int_0^s q(u)du$, for $s \geq 0$. Then $\varphi^*$ is also an Orlicz function and $q$ is its right derivative. $\varphi^*$ is called the function complementary to $\varphi$. Obviously, $\varphi^*$ is the function complementary to $\varphi^*$, i.e., $\varphi^{**} = \varphi$ (see [10, p. 147]).

We say that an Orlicz function $\varphi$ has its complementary function if its derivative $p$ satisfies $p(0) = 0$ and $\lim_{t \to +\infty} p(t) = +\infty$. With the naturally equipped order, an Orlicz sequence space $l_\varphi$ is proved to be a Banach lattice if the Orlicz function $\varphi$ has its complementary function (see Section 3 in [3], or Chapter 2 in [9]). An Orlicz function is said to satisfy the $\Delta_2$-condition (at zero) if there exist $K > 0$ and $t_0 > 0$ such that $\varphi(2t) \leq K\varphi(t)$ for every $0 < t \leq t_0$. These two concepts are not directly used anywhere in our proofs. However, our results are based on techniques introduced in [3] and [9]. And these techniques are only valid when the Orlicz function satisfies the $\Delta_2$-condition and has its complementary function. Therefore these two concepts will still be mentioned in the statement of our results.

We already introduced all the necessary background knowledge and will move to our main results in the next section. Interested readers may check the following references for further study. For the general theory of Orlicz functions, we refer to the book [15], while more detailed information for the Orlicz sequence spaces can be found in book [10].

### 3. Rearrangement Inequality on Positive Tensor Products

In 1934, Hardy, Littlewood and Polya introduced an inequality in their book [7]. Let $\{a_i\}_i$ and $\{b_i\}_i$ be two real sequences both in increasing order. We have
\[ \sum_{i=1}^{\infty} a_i b_{(n+1-i)} \leq \sum_{i=1}^{\infty} a_i b_{(i)} \leq \sum_{i=1}^{\infty} a_i b_i, \]
in which $\rho(i)$ indicates a random permutation. In this section we will construct the similar inequality in both injective and projective tensor products of an Orlicz sequence space with a Banach lattice.

For Banach lattices $X$ and $Y$, let $X\bar{\otimes}Y$ denote the algebraic tensor product of $X$ and $Y$. For each $u = \sum_{k=1}^{\infty} x_k \bar{\otimes} y_k \in X^\ast \bar{\otimes} Y^\ast$, define $T_u : X \rightarrow Y$ by $T_u(x) = \sum_{k=1}^{\infty} x_k^\ast (x) y_k$ for each $x \in X$. The injective cone on $X \bar{\otimes} Y$ is defined by $C_i = \{u \in X \bar{\otimes} Y : T_u(x) \in Y^+ \forall x \in X^+\}$ and the Wittstock injective tensor norm on $X \bar{\otimes} Y$ is defined by
\[ \|u\|_{i} = \inf\{\sup\{\|T_v(x)\| : x \in B_X, v \in C_i, v \pm u \in C_i\} : v \in C_i\}. \]
Let $X\bar{\odot} Y$ denote the completion of $X \bar{\otimes} Y$ with respect to $\|\cdot\|_i$. Then $X\bar{\odot} Y$ with $C_i$ as its positive cone is a Banach lattice (see Wittstock [17, 18] or Meyer-Nieberg [12, Section 3.8]), called the Wittstock injective tensor product of $X$ and $Y$.

Now we consider $l_\varphi \bar{\odot} X$, where $X$ is a Banach lattice and $\varphi$ is an Orlicz function with $\Delta_2$-condition and has its complementary function. We have the following results.

**Lemma 1.** Let $\{a^{(k)}_k\}_k$ and $\{b^{(k)}_k\}_k$ be two sequences in $l_\varphi$ where $\varphi$ is an Orlicz function with $\Delta_2$-condition and has its complementary function, and for any $k$, $a^{(k)} \leq b^{(k)}$. Let $\{x_k\}_k$ and $\{y_k\}_k$ be two sequences in a Banach lattice $X$, and for any $k$, $x_k \leq y_k$. Then
\[ \sum_{k=1}^{m} a^{(k)} \bar{\odot} x_k + \sum_{k=1}^{m} b^{(k)} \bar{\odot} y_k \geq \sum_{k=1}^{m} a^{(k)} \bar{\odot} y_k + \sum_{k=1}^{m} b^{(k)} \bar{\odot} x_k. \]

**Proof.** Let $u_1 = \sum_{k=1}^{m} a^{(k)} \bar{\odot} x_k$, $u_2 = \sum_{k=1}^{m} b^{(k)} \bar{\odot} y_k$, and $u_3 = \sum_{k=1}^{m} a^{(k)} \bar{\odot} y_k$, and...
\( u_4 = \sum_{k=1}^{m} b^{(k)} \otimes x_k. \) We want to prove that \( u_1 + u_2 \geq u_3 + u_4. \)

For each \( u = \sum_{k=1}^{m} s^{(k)} \otimes x_k \in l_p \otimes X, \) where \( s^{(k)} = (s_i^{(k)}), \) define \( (u) = (\sum_{k=1}^{m} s_i^{(k)} x_k) \). In [9, Theorem 3.1], the mapping \( \phi \) has been proved to be an isometry and a Riesz isomorphism. Consider maps \( \phi(u_1 + u_2) = (\sum_{k=1}^{m} a_n^{(k)} x_k + \sum_{k=1}^{m} b_n^{(k)} y_k)_n = (\sum_{k=1}^{m} (a_n^{(k)} x_k + b_n^{(k)} y_k))_n \) and \( \phi(u_3 + u_4) = (\sum_{k=1}^{m} a_n^{(k)} y_k + \sum_{k=1}^{m} b_n^{(k)} x_k)_n = (\sum_{k=1}^{m} (a_n^{(k)} y_k + b_n^{(k)} x_k))_n. \) For any fixed \( n, \)
\[
\sum_{k=1}^{m} (a_n^{(k)} x_k + b_n^{(k)} y_k) - \sum_{k=1}^{m} (a_n^{(k)} y_k + b_n^{(k)} x_k) = \sum_{k=1}^{m} \left( (a_n^{(k)} x_k + b_n^{(k)} y_k) - (a_n^{(k)} y_k + b_n^{(k)} x_k) \right) \geq 0.
\]

That means, \( \phi(u_1 + u_2) \geq \phi(u_3 + u_4). \) We already know that \( \phi \) is an isometry and a Riesz isomorphism in \( l_p \otimes X, \) hence in \( l_p \otimes X. \) Therefore, \( u_1 + u_2 \geq u_3 + u_4. \) □

Let \( A \) denote \( \{a^{(k)}\}_k, \) a sequence in \( l_p, \) and \( B \) denote \( \{x_k\}_k, \) a sequence in a Banach lattice \( X. \) We will then use \( A \otimes B \) to denote \( \sum_{k=1}^{m} a^{(k)} \otimes x_k. \) Using Lemma 1, we can easily derive the Rearrangement Inequality in \( l_p \otimes X. \)

**Theorem 2.** For any \( k, \) let \( A_k \) be a sequence in \( l_p \) where \( \varphi \) is an Orlicz function with \( \Delta_2 \)-condition and has its complementary function, and \( B_k \) be a sequence in a Banach lattice \( X. \) Assume that sequences \( \{A_k\}_k \) and \( \{B_k\}_k \) are both in increasing order. Then \( \sum_{i=1}^{m} A_i \otimes B_{(m+1-i)} \leq \sum_{i=1}^{m} A_i \otimes B_{\rho(i)} \leq \sum_{i=1}^{m} A_i \otimes B_i, \) in which \( \rho(i) \) indicates a random permutation.

Similar results can also be derived in projective tensor product. For Banach lattices \( X \) and \( Y, \) the projective cone on the tensor product \( X \otimes Y \) is defined to be \( C_p = \{\sum_{k=1}^{n} x_k \otimes y_k : n \in \mathbb{N}, x_k \in X^+, y_k \in Y^+\}. \)

Fremlin [5, 6] introduced the positive projective tensor norm on \( X \otimes Y \) as follows:
\[
\|u\|_{p1} = \sup\{\|\sum_{k=1}^{n} x_k \otimes y_k\| : u = \sum_{k=1}^{n} x_k \otimes y_k, \psi \in M\},
\]
where \( M \) is the set of all positive bilinear functionals on \( X \times Y \) with their norms less than or equal to 1.

Let \( X \otimes_F Y \) denote the completion of \( X \otimes Y \) with respect to \( \|\|_{p1}. \) Then \( X \otimes_F Y \) with \( C_p \) as its positive cone is a Banach lattice (see Fremlin [5, 6]), called the Fremlin projective tensor product.

Fremlin also introduced a famous theorem, which will be used in the proof of our next result.

**Fremlin’s Theorem.** Let \( X \) and \( Y \) be Banach lattices. Then for every Banach lattices \( G \) and every bipositive map \( T : X \times Y \rightarrow G, \) there exists a unique positive map \( T^\oplus : X \otimes_F Y \rightarrow G \) such that \( \|T^\oplus\| = \|T\| \) and \( T^\oplus(x \otimes y) = T(x, y) \) for all \( x \in X \) and \( y \in Y. \)

Moreover, \( T^\oplus \) is a Riesz homomorphism if and only if \( T \) is a Riesz bimorphism.

We now put our focus on \( l_p \otimes_F X, \) where \( X \) is a Banach lattice and \( \varphi \) is an Orlicz function with \( \Delta_2 \)-condition and has its complementary function.

**Lemma 3.** Let \( a = (a_n)_n \) and \( b = (a_n)_n \) be two elements in \( l_p, \) where \( \varphi \) is an Orlicz function with \( \Delta_2 \)-condition and has its complementary function. Let \( x \) and \( y \) be two elements in a Banach lattice \( X, \) and \( x \leq y. \) Then \( a \otimes_F x + b \otimes_F y \geq a \otimes_F y + b \otimes_F x. \)

**Proof.** For all \( s = (s_n)_n \in l_p \) and all \( x \in X, \) define \( T \) from \( l_p \times X \) to a sequential Banach lattice by \( T(s,x) = (s_n x)_n. \) Applying Fremlin’s Theorem, Lai already proved that \( T(s,x) = T^\oplus(s \otimes x), \) and \( T^\oplus \) (hence \( T \)) is an isometry and a Riesz isomorphism (see [9], Theorem 4.1).

Consider \( T^\oplus(a \otimes_F x + b \otimes_F y) = T^\oplus(a \otimes_F x) + T^\oplus(b \otimes_F y) = T(a, x) + T(b, y) = (a_n x)_n + (b_n y)_n = (a_n x + b_n y)_n, \) and \( T^\oplus(a \otimes_F y + \)
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\[ b \otimes_F x = T^\otimes (a \otimes_F y) + T^\otimes (b \otimes_F x) = T(a, y) + T(b, x) = (a_n y)_n + (b_n x)_n = (a_n y + b_n x)_n . \]

For any \( n \), \( a_n x + b_n y \geq a_n y + b_n x \) according to the original Hardy-Littlewood-Polya Rearrangement Inequality, \( (a_n + b_n)_n \geq ((a_n y + b_n x)_n \). Therefore \( T^\otimes (a \otimes_F x + b \otimes_F y) \geq T^\otimes (a \otimes_F y + b \otimes_F x) \). Because \( T^\otimes \) is an isometry and a Riesz isomorphism, we can conclude that \( a \otimes_F x + b \otimes_F y \geq a \otimes_F x + b \otimes_F x \). \( \Box \)

As a direct generalization of Lemma 3, the next result can be easily derived.

**Theorem 4.** Let \( \{a_k\}_k \) be a sequence in \( l_p \) where \( p \) is an Orlicz function with \( \Delta_2 \)-condition and has its complementary function. Let \( \{x_k\}_k \) be a sequence in a Banach lattice \( X \). Assume that sequences \( \{a_k\}_k \) and \( \{x_k\}_k \) are both in increasing order. That is, \( a_k \leq a_{k+1} \) and \( x_k \leq x_{k+1} \) for any \( k \). Then
\[
\sum_{i=1}^n a_i \otimes_F x_{(m+1-i)} \leq \sum_{i=1}^n a_i \otimes_F x_{\rho(i)} \leq \sum_{i=1}^n a_i \otimes_F x_i,
\]

where \( \rho \) indicates a random permutation.

**4. Chebyshev’s Sum Inequality on Positive Tensor Products**

The Chebyshev’s Sum Inequality states, for any two real number sequences \( \{a_i\}_i \) and \( \{b_i\}_i \), both in increasing (or decreasing) order,
\[
\frac{1}{n} \sum_{i=1}^n a_i b_i \geq \left( \frac{1}{n} \sum_{i=1}^n a_i \right) \left( \frac{1}{n} \sum_{i=1}^n b_i \right).
\]

In this section I will discuss a similar inequality in both positive tensor products.

There are many ways to prove the Chebyshev’s Sum Inequality. The method I would like to focus on is the one using rearrangement inequality. Let \( A_i = a_1 b_1 + \cdots + a_n b_n \) for all \( i \). Therefore, \( \sum_{i=1}^n A_i = n \sum_{i=1}^n a_i b_i \). Now let \( B_1 = a_1 b_1 + \cdots + a_n b_n \), \( B_2 = a_1 b_2 + \cdots + a_n b_1 \), \( B_3 = a_1 b_3 + \cdots + a_n b_{n-1} + a_n b_1 \), \( B_4 = a_1 b_4 + \cdots + a_n b_{n-2} + a_n b_2 + a_1 b_1 \), \( \cdots \), \( B_n = a_1 b_n + a_2 b_1 + \cdots + a_n b_{n-1} \). Then \( \sum_{i=1}^n B_i = \sum_{i=1}^n A_i = \sum_{i=1}^n a_i \sum_{i=1}^n b_i \). We already know that \( A_i \geq B_i \) for all \( i = 1, \cdots, n \), according to the rearrangement inequality. Since \( n \sum_{i=1}^n a_i b_i = \sum_{i=1}^n A_i \geq \sum_{i=1}^n B_i = (\sum_{i=1}^n a_i)(\sum_{i=1}^n b_i) \), we then have the Chebyshev’s Sum Inequality by multiplying \( \frac{1}{n} \) to both sides.

We can see that the rearrangement inequality plays a key role in the above proof. And since we already have the rearrangement inequality in both \( l_p \otimes X \) and \( l_p \otimes_F X \), we therefore can derive the Chebyshev’s Sum Inequality in both tensor products using the same technique. The detail of the proofs, almost identical to the proof mentioned earlier, is left to the readers.

**Theorem 5.** For any \( k \), let \( A_k \) be a sequence in \( l_p \) where \( p \) is an Orlicz function with \( \Delta_2 \)-condition and has its complementary function, and \( B_k \) be a sequence in a Banach lattice \( X \). Assume that sequences \( \{A_k\}_k \) and \( \{B_k\}_k \) are both in increasing or decreasing order. Then
\[
\frac{1}{n} \sum_{k=1}^n A_k \otimes F B_k \geq \left( \frac{1}{n} \sum_{k=1}^n A_k \right) \otimes F \left( \frac{1}{n} \sum_{k=1}^n B_k \right).
\]

The equality holds if \( A_i = A_j \) or \( B_i = B_j \) for any \( i \neq j \).

**Theorem 6.** For any \( k \), let \( \{x_k\}_k \) be a sequence in \( l_p \) where \( p \) is an Orlicz function with \( \Delta_2 \)-condition and has its complementary function \( p^* \), and \( \{y_k\}_k \) be a sequence in a Banach lattice \( X \). Assume that sequences \( \{x_k\}_k \) and \( \{y_k\}_k \) are both in increasing or decreasing order. Then
\[
\frac{1}{n} \sum_{k=1}^n x_k \otimes F y_k \geq \left( \frac{1}{n} \sum_{k=1}^n a_k \right) \otimes F \left( \frac{1}{n} \sum_{k=1}^n b_k \right).
\]

The equality holds if \( x_i = x_j \) or \( y_i = y_j \) for any \( i \neq j \).

**References**


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(2008), 45-54.


