Random Matrix Theory Based Cognitive Radio Spectrum Sensing

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Abstract: A cognitive radio system allows higher data transmission rates due to the efficient spectrum utilization. Spectrum sensing plays a substantial role in such a cognition scenario. In this paper, a novel multiple antenna sensing algorithm is proposed for detecting the presence or the absence of the primary user signal. The scheme is called CRABWISE (Cognitive RAdio sensing Based on the joint distribution of pseudo Wishart matrix Eigenvalues). It turns out that without prior information about the PU (primary user) signal, CRABWISE performs near to the optimal sensing performance, which is observed for an energy detection sensing being equipped with perfect prior information of the PU signal. The performance of CRABWISE is investigated using the receiver operating characteristic for signals transmitted over a delay-dispersive channel. Moreover, we study how to find the optimum threshold for the proposed test numerically. The achievable performance is considered for increasing length of the received signal frame in terms of both probability of detection and probability of a wrong decision.

Key words: Pseudo Wishart matrix, Bayesian test, likelihood ratio, receiver operating characteristic.

1. Introduction

To meet the ever-increasing demand on high data rates expected by the International Telecommunication Union [1] especially for mobile communication applications, it is very important to fully and efficiently utilize the available resources. The problem of spectral holes or idle bands is reported in Ref. [2] showing that many frequency bands dedicated to different applications are idle.

CR (cognitive radio) systems [3] come as a solution to improve the efficiency of spectrum utilization by assigning some users higher priority in exploiting the spectrum than other users. The prioritized user is the so-called PU (primary user), while the user with lower priority is called SU (secondary user) being allowed to use the spectrum after having checked the spectrum availability. The classification of the users facilitates the opportunistic transmission by SUs in an efficient manner.

Although there is no clear-cut definition for CR system, there are some key components in this system. The latter comprise, for instance, spectrum sensing or exploring idle bands, spectrum sharing schemes to allow the sharing without interfering on PU signals, location identification and system discovery [4]. Spectrum sensing is considered as the fundamental component for offering efficient spectrum utilization.

In general, spectrum sensing techniques can be classified into different categories. These are ED (energy detection) based sensing, waveform based sensing, cyclostationarity based sensing and EB (eigenvalue-based) sensing. No need for prior information about the PU signal is required in ED, which also has a low implementation complexity [5-7]. On the other hand, it performs poorly either in the low SNR (signal-to-noise ratio) regime [8] or when the estimation of the noise variance is inaccurate [9]. ED can achieve a close to optimum performance when the noise power and signal distributions under different
Random Matrix Theory Based Cognitive Radio Spectrum Sensing

Signal hypotheses are perfectly known. Concerning waveform or matched filtering based sensing, it outperforms the ED at the expense of using a known predefined and synchronized signaling waveform [10]. Cyclostationarity based sensing exploits the periodicity of the received signal to sense the spectrum. Here, signal statistics have to be estimated [11-13].

EB sensing can outperform the other techniques without prior knowledge of the PU signal and shows higher robustness against noise power level uncertainty [14]. Lately, three new multi-antenna EB algorithms based on advances in RMT (random matrix theory) [15, 16] have been introduced, namely MME (maximum-minimum eigenvalue), EME (energy with minimum eigenvalue) and MED (maximum eigenvalue detection) [17]. In these three algorithms, different test statistics have been applied. The MME depends on the ratio of the maximum and the minimum eigenvalues of the covariance matrix of the received frame. The EME adopts the ratio of the average power of the received signal and the minimum eigenvalue as test statistic. The MED makes use of largest eigenvalue to sense the spectrum. All three schemes result from recently derived distributions for the aforementioned ratios. However, the joint PDF (probability density function) of the covariance matrix eigenvalues has never been considered properly for spectrum sensing due to the difficulty associated with the joint PDF of the eigenvalues [17].

In this letter, we devise a multi-antenna sensing algorithm, which relies on the joint statistics of eigenvalues of the received signal. RMT and the study of the eigenvalue distributions are well-known techniques in physics and statistics. In addition, it has been applied for different purposes in communication systems [15], e.g., for deriving capacity expressions for MIMO (multiple-input multiple-output) systems [18].

The proposed scheme considered here builds on the CRABWISE (Cognitive RAdio sensing Based on the joint distribution of pseudo WIShart matrix Eigenvalues) algorithm presented for the first time in Ref. [19]. It makes use of the derived joint PDF of eigenvalues of pseudo Wishart matrices for both correlated and uncorrelated signal frames to devise a hypothesis test for spectrum sensing. Assuming the received signal frame is $X$, the joint PDFs discussed in Refs. [20, 21] are exploited in CRABWISE to create a test statistic relying on the joint PDF of the eigenvalues of $XX^\dagger$ to detect the presence of a PU. Here, we will present additional results complementing the ones in Ref. [19].

The rest of this paper is organized as follows. Section 2 describes the system model. Section 3 presents the complete derivation of CRABWISE. A practical implementation of the latter is proposed in Section 4. The performance is investigated in Section 5. Finally, conclusions are drawn in Section 6.

Throughout the paper, a boldface quantity like $X = \{X_{n,m}\}$ defines a random matrix of suitable dimension, where $X_{n,m}$ is a random variable in the $n$th row and $m$th column and $x$ denotes an instance of $X$. The transpose and the Hermitian transpose of $X$ are written as $X^T$ and $X^\dagger$, respectively. Furthermore, $I_N$, $\| \|$ and $E\{ \}$ denote the $(N \times N)$ identity matrix, the determinantal the expectation, respectively.

2. System Model

Assume a SU receiver equipped with a number of $N$ antennas receiving a band-limited vector signal modeled as a random vector process. The latter is sampled with Nyquist rate to provide the matrix signal $X$ of dimension $(N \times M)$ with $N \leq M$. The SU receiver aims to detect the presence of a PU signal based on $X$. The elements of $X$ represent either samples of a PU signal embedded in AWGN (additive white Gaussian noise) or AWGN only. The $n$th row of $X$ represents the samples taken at the $n$th antenna.
We model $\mathbf{X}$ in the complex baseband under both hypotheses as follows.

2.1 PU Signal Absent (Hypothesis $H_0$)

This is the case where there is no PU and the $m$th sample, $m = 1, \ldots, M$, in the $n$th row of $\mathbf{X}$, $n = 1, \ldots, N$ is modeled by

$$X_{n,m} = W_{n,m}$$

The noise vector $\mathbf{W}_n = [W_{n,1}, \ldots, W_{n,M}]^T$ is an $M$-variate circularly-symmetric complex normal random vector with the same covariance matrix $\mathbb{E} \{ \mathbf{W}_n \mathbf{W}_n^H \} = \mathbf{I}_M$ for all $n$. Furthermore, we assume the noise signals with unit variance to be mutually independent at two different antennas.

2.2 PU Signal Present (Hypothesis $H_1$)

There is a PU signal present and we model the received signal by

$$X_{n,m} = S_{n,m} + W_{n,m}$$

The PU signal vector $\mathbf{S}_n = [S_{n,1}, \ldots, S_{n,M}]^T$ has zero-mean with covariance matrix

$$\Sigma = \mathbb{E} \{ \mathbf{S}_n \mathbf{S}_n^H \}$$

$\Sigma$ is assumed to be identical for all $n$ and known at the receiver with diagonal elements $\sigma_X^2$ being identical and positive. Due to the unit variance of the noise samples, the SNR is defined by $\gamma = \sigma_X^2$.

The signals at two different antennas are considered mutually independent. Moreover, it is straightforward to show that the output $S_{n,m}$ of the $n$th channel assumed to be uncorrelated scattering with a wide-sense stationary transmitted signal input is stationary in $m$. Consequently, $\Sigma$ is a Toeplitz matrix. Finally, we assume any signal $S_{n,m}$ to be independent of any noise signal $W_{p,l}$ for $n, p = 1, \ldots, N$ and $m, l = 1, \ldots, M$.

3. CRABWISE Algorithm

The objective of the CRABWISE approach is to devise an algorithm for spectrum sensing. The latter requires to detect a PU signal independently of and without prior knowledge of the specific nature of the PU signal at hand. In view of the transmission of a PU signal over a delay-dispersive medium with uncorrelated scattering, we invoke the central limit theorem to model $\mathbf{S}_n$ as multivariate Gaussian distributed with a covariance matrix defined in Eq. (3).

Similarly, we can define the vectors $\mathbf{X}_n = [X_{n,1}, \ldots, X_{n,M}]^T$ containing the elements of the row vectors of $\mathbf{X}$. According to the assumptions made in Section 2, the covariance matrix $\Sigma_{pw} = \mathbb{E} \{ \mathbf{X}_n \mathbf{X}_n^H \}$ under both hypotheses is obviously independent of $n$, and $\mathbf{X}_n$ and $\mathbf{X}_p$ are independent for $n \neq p$. Therefore, the matrix $\bar{\mathbf{X}} = \mathbf{X}^H / \sigma_X^2$ of dimension $(M \times M)$ is central pseudo Wishart matrix and

$$\bar{\mathbf{X}} \sim \mathcal{PW}_N(M, \Sigma_{pw})$$

where $\sigma_X^2$ is the variance of the elements of $\mathbf{X}$.

Two different cases will be distinguished in formulating the hypothesis testing based on $\Sigma_{pw}$, namely an uncorrelated central pseudo Wishart case or a correlated central pseudo Wishart case [21, 22]. As shown in Ref. [22], the matrix $\mathbf{X}^H / \sigma_X^2$ has $M$ eigenvalues $N$ of which are identical to the eigenvalues of the $(N \times N)$ matrix $\mathbf{XX}^H / \sigma_X^2$ while the residual $M - N$ eigenvalues are zero-valued.

As a result, we focus on the joint PDF of the $N$ nonzero eigenvalues of $\bar{\mathbf{X}}$ in devising the hypothesis test rather than using all $M$ eigenvalues.

Under hypothesis $H_0$ the matrix $\bar{\mathbf{X}}$ follows an uncorrelated central pseudo Wishart distribution with $\bar{\mathbf{X}} \sim \mathcal{PW}_N(M, \mathbf{I}_M)$. The $N$-dimensional vector of the ordered eigenvalues of $\bar{\mathbf{X}}$ in decreasing order is defined by

$$\Lambda_{\bar{X}} = \Lambda(\bar{\mathbf{X}})$$

where, $\Lambda(\cdot)$ denotes the corresponding eigenvalue
vector operator of the argument. Based on the identity of $N$ eigenvalues of $X^\dagger X/\sigma_x^2$ and $XX^\dagger /\sigma_x^2$, the joint PDF of $\mathbf{A}_X$ can be written as [20]

$$p_{A_X}(\lambda | H_o) = |v(\lambda)| \prod_{i=1}^N \frac{\exp(-\lambda_i)\lambda_i^{M-N}}{(N-i)!(M-i)!}$$ \hspace{1cm} (5)

where, $\lambda = [\lambda_1, \ldots, \lambda_N] \in \mathbb{R}^N$ is an instance of $\mathbf{A}_X$ and $v(\lambda)$ is an $(N \times N)$ Vandermonde matrix with elements in the $p$th row and the $q$th column given by

$$v_{p,q} = \lambda_p^{q-1}$$

where, $1 \leq p,q \leq N$. The determinant expression in Eq. (5) can be written as [22]

$$|v(\lambda)| = \prod_{i=1}^N (\lambda_i - \lambda_j)$$ \hspace{1cm} (6)

Under hypothesis $H_1$ the matrix $\tilde{X}$ follows an uncorrelated central pseudo Wishart distribution with $\tilde{X} \sim \mathcal{PW}_N(M,(\Sigma + I_M)/((1+\sigma_x^2)))$, where $\Sigma$ is the covariance matrix described in Section 2. In this case the joint PDF of $\mathbf{A}_X$ can be written as [21]

$$p_{A_X}(\lambda | H_1) = |g(\lambda)| \prod_{i=1}^N \frac{1}{(N-i)!} \prod_{i=1,j<\lambda}^M \frac{1}{\sigma_j - \sigma_i}$$ \hspace{1cm} (7)

The elements of the matrix $|g(\lambda)|$ are given by

$$g_{p,q} = \begin{cases} 
\sigma_p^{q-1} & \text{for } 1 \leq q \leq M-N \\
\sigma_p^{M-N-1}\exp(-\frac{\lambda_p^{M}M+N}{\sigma_p}) & \text{for } M-N \leq q \leq M 
\end{cases}$$ \hspace{1cm} (8)

We denote the ordered eigenvalues of $(\Sigma + I_M)/((1+\sigma_x^2))$ by $\sigma_1 > \sigma_2 > \cdots > \sigma_M$. Using Eq. (5) and Eq. (7), we can describe the likelihood threshold by

$$L(\lambda) = \frac{p_{A_X}(\lambda | H_1)}{p_{A_X}(\lambda | H_0)}$$

$$= \frac{|g(\lambda)| \prod_{i=1}^N (M-i)!}{|v(\lambda)| \prod_{i=1,j\neq i}^M (\sigma_j - \sigma_i) \prod_{i=1}^N \exp(-\lambda_i)\lambda_i^{M-N}}$$ \hspace{1cm} (9)

The last equation results from substituting $|v(\lambda)|$ using Eq. (6) and $\Gamma_N(M) = \prod_{i=1}^N (M-i)!$.

4. Practical Implementation of CRABWISE

The derived Bayesian test in Section 3 minimizes the Bayes risk so it is the optimum test for the assumption made in the system model. Unfortunately, a practical implementation of this test is associated with many difficulties due to a number of reasons. First, to reliably sense the spectrum, any algorithm needs a large observation length, i.e., a large value of $M$ in CRABWISE. Choosing $M$ to be large makes the dimension of $\tilde{X}$ high, which in turn renders the calculation of the eigenvalues highly complex. Secondly, a floating-point accuracy is needed to avoid numerical effects in calculating the determinants $|g(\lambda)|$ and $|v(\lambda)|$ when $M$ is large. This kind of accuracy is not always available in practical systems.

Hence, a practical solution is needed for approximating the likelihood ratio in Eq. (10) when $M$ is large. To that end, we propose an alternative solution relying on dividing the received observation frame of length dimension $(N \times M)$ into number of smaller subframes with dimension $(N \times M_\pm)$. The likelihood ratios of the subframes will be averaged to
find the sought for approximation of Eq. (10).

The practical implementation can be applied with the following steps:

1. Based on a received frame $X$, we calculate $\hat{X} = X'X / \sigma^2_X$, which follows a pseudo Wishart distribution $\hat{X} \sim \mathcal{W}_N(M, \Sigma_{pw})$ according to the requirements discussed in Section 2.

2. After normalization, the frame $X$ is divided into $K$ subframes $\Xi^{(1)}, \Xi^{(2)}, \ldots, \Xi^{(K)}$ of dimension $(N \times M_z)$ as shown in Fig. 1 here $K = M / M_z \in \mathbb{N}$. As argued in Section 3, $M_z$ should be chosen sufficiently small to apply the test without numerical problems. Moreover, the selection of $M_z$ must be such that $N \leq M_z$ as discussed in Section 2.

3. We calculate for each of the resulting subframes $\tilde{\Xi}^{(k)} = \tilde{\Xi}^{(k)}_1 \tilde{\Xi}^{(k)}_2 \ldots \tilde{\Xi}^{(k)}_N$ where $k = 1, \ldots, K$. Afterwards, the likelihood ratio $L_k(\lambda)$ of the $k$th subframe is calculated based on the $N$-dimensional non-zero ordered eigenvalue vector of $\tilde{\Xi}^{(k)}$ denoted by $\Lambda_{\tilde{\Xi}^{(k)}}(\tilde{\Xi}^{(k)})$. Based on Eq. (10), the likelihood ratio of the $k$th subframe reads

$$L_k(\lambda) = \frac{\beta(\lambda)^{1/2}}{\prod_{i=1}^{N} (\lambda_j - \lambda_i)} \prod_{i=1}^{N} e^{-\lambda_i} \lambda_i^{N/2}$$

where, $\lambda = [\lambda_1, \ldots, \lambda_N] \in \mathbb{R}^N$ is an instance of $\Lambda_{\tilde{\Xi}^{(k)}}(\tilde{\Xi}^{(k)})$.

4. We use the average of all subframes’ likelihood ratio for approximating the likelihood ratio of the frame $X$ according to

$$L(\lambda) = \frac{1}{K} \sum_{k=1}^{K} L_k(\lambda) = L(\lambda)$$

The latter will be used later on as test statistics in the practical implementation of CRABWISE with the decision regions discussed in Section 3.

5. The threshold $\tau$ of the test depends generally on prior probabilities, cost assignments and many other parameters. Instead of choosing these quantities, we numerically optimize $\tau$ to find the optimum threshold $\tau_{opt}$ which minimizes the probability of a misdetection. The optimization procedure is described in Section 5.

5. Performance Analysis

In this section, the performance of the proposed algorithm is investigated and compared to another classical sensing technique, namely the ED based sensing. We assume prior probabilities of both hypotheses $\pi_0 = \pi_1 = 0.5$. The values of the frame parameters are $N = 5$, $M = 600$ and the number of subframes within one frame is $K = 100$ and thus $M_z = 6$. The PU signal $S_{n,m}$ is simulated as stream of temporally correlated BPSK (binary phase-shift keying) symbols. Note that choosing a BPSK signal is violating the required Gaussian distribution of the matrix elements $S_{n,m}$.

The covariance matrix of the correlated stream is

$$\Sigma = \begin{bmatrix} 1 & 0.6 & 0.59 & 0.58 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.59 & 0.58 & 0 \\ 0.59 & 0.6 & 1 & 0.6 & 0.59 & 0.58 \\ 0.58 & 0.59 & 0.6 & 1 & 0.6 & 0.59 \\ 0 & 0.58 & 0.59 & 0.6 & 1 & 0.6 \end{bmatrix}$$

Such a matrix could be considered the covariance matrix of a BPSK signal at the output of a delay-dispersive channel.
5.1 Finding the Optimum Threshold $\tau_{\text{opt}}$

The objective of CRABWISE is to reliably detect the presence or absence of a PU signal. Therefore, we optimize the threshold where the objective is to minimize the probability of a wrong decision or probability of error given by $P_E$, where the wrong decision is either a false alarm under hypothesis $H_0$ or a missed detection under hypothesis $H_1$. The probability of error can be described in case of equally probable hypotheses as the arithmetic mean of the probability of false alarm $P_{FA}$ and the probability of missed detection $P_{MD} = 1 - P_D$.

With $P_E = \frac{1}{2}(P_{FA} + P_{MD})$, Fig. 2 shows $P_E$ for different values of the threshold $\tau$. Clearly, changing the threshold will lead to different values of $P_E$.

Similarly, a specific threshold will show varying sensing performance for varying SNR values as shown in Fig. 3. In other words, the optimum threshold depends on the value $\gamma$.

In general, increasing the threshold reduces $P_{FA}$ at the expense of a higher $P_D$. Fig. 4 shows the procedure of optimizing the threshold for $\gamma = -5 \text{ dB}$. The minimum value of $P_E$ is achieved for $\tau_{\text{opt}} = 92$. In fact, the optimum threshold does not depend solely on $\gamma$, but also on the system parameters $M$, $N$, $M_e$, $\pi_0$, $\pi_1$ and the cost assignment.

In Fig. 4, the optimum threshold is shown to be different for different values of $\gamma$. This motivates finding the optimum thresholds at different $\gamma$ values numerically. This is accomplished in Fig. 5. after defining the system parameters. Then, we optimize $\tau$ as a function of $\gamma$.

The resulting function $\tau_{\text{opt}} = \tau_{\text{opt}}(\gamma)$ in Fig. 5 is clearly an increasing function for increasing $\gamma$.

5.2 Receiver Operating Characteristic

Here, the performance of CRABWISE is assessed using the ROC (receiver operating characteristics), which show $P_D$ as a function of $P_{FA}$ [23] for different threshold values $\tau$. According to Ref. [24], when the receiver has perfect knowledge of noise statistics and the received signal under both hypotheses,

![Fig. 2 $P_E$ for using different thresholds $\tau$.](image-url)
the ED performs near to optimum. In fact, this explains the performance achieved by CRABWISE, since it also achieves near-optimum performance as the comparison in Fig. 6 shows for two different $\gamma$ values. The proposed algorithm has no information about the received signal except for the knowledge of $\Sigma$, which is neither complex nor difficult to be estimated. It is also clear in Fig. 6 that CRABWISE
Fig. 5  Optimum threshold $\tau_{\text{opt}} = \tau_{\text{opt}}(\gamma)$.

Fig. 6  ROC comparison of ED and CRABWISE for $\gamma = -5$ dB and $\gamma = -10$ dB.

leads to a reliable sensing for $\gamma \geq -5$ dB.

5.3 Increasing the Observation Length $M$

Here, we keep all the aforementioned parameters the same except for varying $M$ for a fixed SNR value. The objective is to show the impact of an increasing observation length on the achievable performance. Put differently, we want to quantize the
required frame length to achieve a specific performance level. Fig. 7 illustrates the minimum probability of a wrong decision for increasing $M$ together with the optimum threshold $\tau_{\text{opt}}$.

As expected, the performance improves for increasing number of observations. Moreover, the optimum threshold $\tau_{\text{opt}}$ is also changing for increasing $M$ as discussed above.

Note that we have only considered optimizing the threshold with respect to the SNR rather than with regard to other parameters.

In some scenarios, the considered system is subject

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{$P_{E_{\text{min}}}$ and $\tau_{\text{opt}}$ at $\gamma = -10$ dB for increasing $M$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8}
\caption{Maximally achievable $P_D$ for $P_{\text{FA}} = 0.05$ at $\gamma = -10$ dB and increasing $M$.}
\end{figure}
to a specific maximum value of $P_{FA}$, therefore, it is important to know the observation length required to achieve a certain $P_D$. In Fig. 8, the maximum achievable $P_D$ subject to $P_{FA} = 0.05$ is shown for $\gamma = -10$ dB as a function of the observation length $M$.

6. Conclusions

CRABWISE outperforms classical sensing techniques in cognitive radio by applying random matrix theory. CRABWISE shows higher robustness against changes in noise power as an advantage over energy detection based sensing, where the performance is highly dependent on the noise power uncertainty. It turns out that CRABWISE leads to a performance close to the optimal detector, however, at the cost of extra complexity in terms of signal processing. The choice of the threshold after splitting the frame into a number of subframes can lead to different performance levels. Changing the threshold can affect both the probability of detection and the probability of false alarm making CRABWISE more adaptive to different scenarios. Moreover, the optimum threshold can be numerically optimized as a function of the SNR where the objective is to minimize the probability of a wrong decision. For BPSK transmission by a primary user over a delay-dispersive channel, CRABWISE achieves reliable detection for an SNR value of $\gamma = -5$ dB.

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